3.1 Derivatives of Polynomials and Exponentials

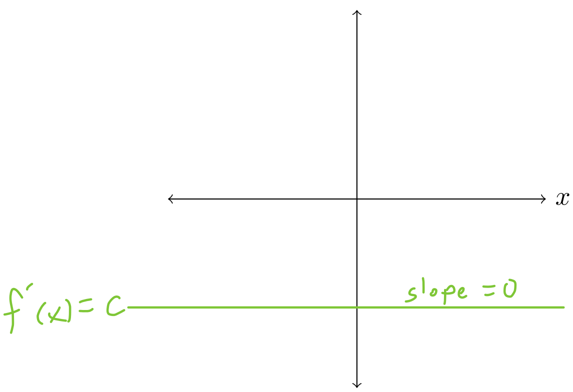
Learning Objectives: After completing this section, we should be able to

- find the derivative of a constant function using the definition of a derivative.
- derive the derivative of a power function with an integral exponent.
- derive the Constant Multiple Rule, the Sum Rule, and the Difference Rule.
- apply the general Power Rule to find the derivative of a power function with a real-valued exponent.

3.1.1 Polynomials

Evaluating $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ is tedious... but instructive! Let's establish some shortcuts, noting that all of them come from the definition.

- **Constant Rule:**



Let $f(x) = c$, for any constant c.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

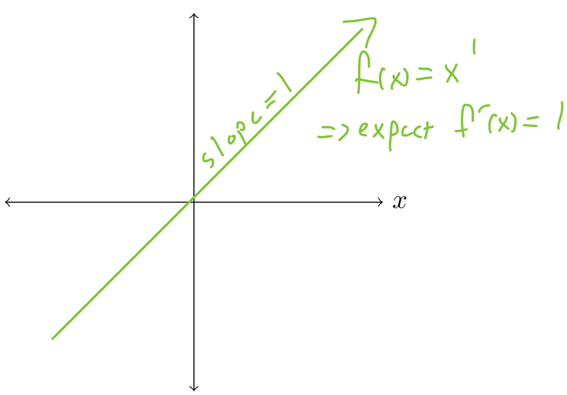
$$= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

Rule: $\frac{d}{dx}\, c = 0$, for any constant c.

- **Power Rule:**



$f(x) = x^p$

(a) Assume $p=1$; i.e., $f(x) = x^1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{(x}+h) - \cancel{x}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

(b) Let $f(x) = x^p$, where $p = 2, 3, 4, \ldots$

For any input a,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \frac{x^p - a^p}{x-a}$$

$$= \lim_{x\to a} \frac{\cancel{(x-a)}\left[x^{p-1} + a x^{p-2} + a^2 x^{p-3} + \cdots + a^{p-2}\cdot x + a^{p-1}\right]}{\cancel{x-a}}$$

(Note, this is the Binomial Theorem, and there are p terms in purple)

Power rule continued:

$$= \lim_{x \to a} \left(x^{p-1} + a x^{p-2} + \cdots + a^{p-2} \cdot x + a^{p-1}\right)$$

$$= a^{p-1} + a \cdot a^{p-2} + \cdots + a^{p-2} \cdot a + a^{p-1}$$

$$= \underbrace{a^{p-1} + a^{p-1} + \cdots + a^{p-1} + a^{p-1}}_{p \text{ terms}}$$

$$= p\left(a^{p-1}\right)$$

So, we've shown $f'(a) = p \cdot a^{p-1}$ for any input a

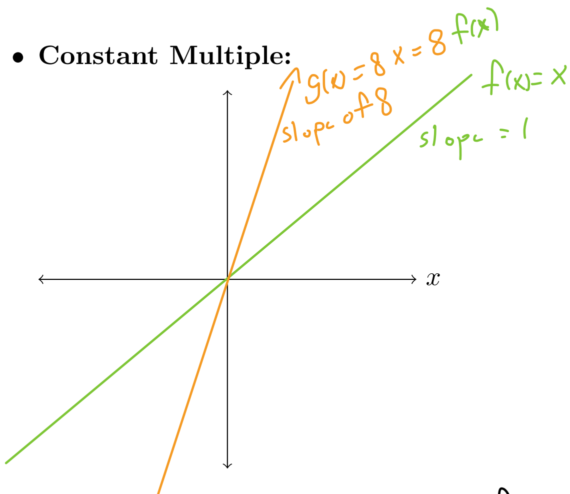
$$\Rightarrow \quad f'(x) = p\, x^{p-1}$$

Rule: $\frac{d}{dx} x^p = p \cdot x^{p-1}$, for any real number p

(The above proof only works for p an integer)

Ex) $f(x) = x^{\pi} \Rightarrow f'(x) = \pi \cdot x^{\pi - 1}$

- **Constant Multiple:**



Let $g(x) = c \cdot f(x)$, for any constant c and f is any differentiable fnctn.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h \to 0} c\left(\frac{f(x+h) - f(x)}{h}\right)$$

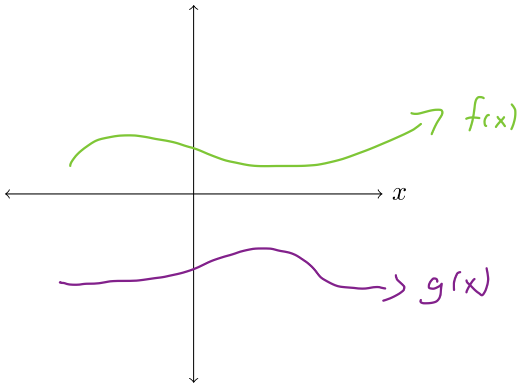
$$= c\left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right)$$

$$= c \cdot f'(x).$$

Rule: $\frac{d}{dx}\left(c \cdot f(x)\right) = c \cdot f'(x)$, where c is any constant and f is differentiable.

- **Sums:**

slope of a sum is the sum of slopes



Let $s(x) = f(x) + g(x)$, where f & g are differentiable.

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$, where f & g are differentiable.

Example. Find the derivative of $f(x)$.

1. $f(x) = x^{12}$ (here $p = 12$)

$$f'(x) = p x^{p-1} = 12 \cdot x^{12-1} = 12x^{11}$$

2. $f(x) = 2x^{12}$ (constant multiple of 2)

$$f'(x) = \frac{d}{dx}(2x^{12}) = 2\left(\frac{d}{dx} x^{12}\right) = 2(12x^{11}) = 24x^{11}$$

3. $f(x) = 2^{12} = 4096$

Derivative of a constant!

$$f'(x) = 0.$$

4. $f(x) = \frac{-7}{8}x^3$

$$f'(x) = \frac{d}{dx}\left(-\frac{7}{8}x^3\right) = -\frac{7}{8}\left(\frac{d}{dx}x^3\right)$$

$$= -\frac{7}{8}(3 \cdot x^{3-1})$$

$$= -\frac{21}{8}x^2$$

You try!

5. $f(x) = 5x^3 + 3x + 1$

$$f'(x) = \frac{d}{dx}\left[5x^3 + 3x + 1\right] = \frac{d}{dx}(5x^3) + \frac{d}{dx}(3x) + \frac{d}{dx}(1)$$
$$= 5\left(\frac{d}{dx}x^3\right) + 3\left(\frac{d}{dx}x^1\right) + 0$$
$$= 5(3x^{3-1}) + 3\cdot 1\cdot x^{1-1} + 0$$
$$= 15x^2 + 3$$

$$\left(3\cdot 1\cdot x^{1-1} = 3\cdot 1\cdot x^0 = 3\cdot 1\cdot 1 = 3\right)$$

6. $f(x) = 3x^{-5}$ $= 3\frac{1}{x^5} = \frac{3}{x^5}$

$$f'(x) = 3\cdot(-5)\,x^{-5-1}$$
$$= -15\,x^{-6}$$

You try!

7. $f(x) = \pi x^\pi$

$$f'(x) = \frac{d}{dx}(\pi x^\pi)$$
$$= \pi\frac{d}{dx}x^\pi$$
$$= \pi\cdot\pi\cdot x^{\pi-1}$$
$$= \pi^2 x^{\pi-1}$$

8. $f(x) = \frac{\pi}{2}\sqrt{x}$

$$= \frac{\pi}{2}x^{\frac{1}{2}}$$

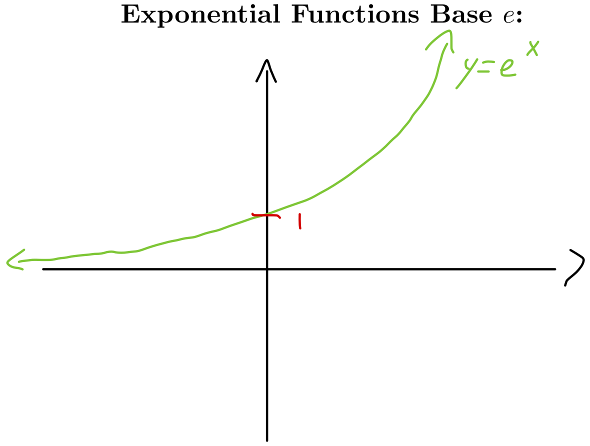
$$f'(x) = \frac{\pi}{2}\cdot\frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{\pi}{4}x^{-\frac{1}{2}}$$

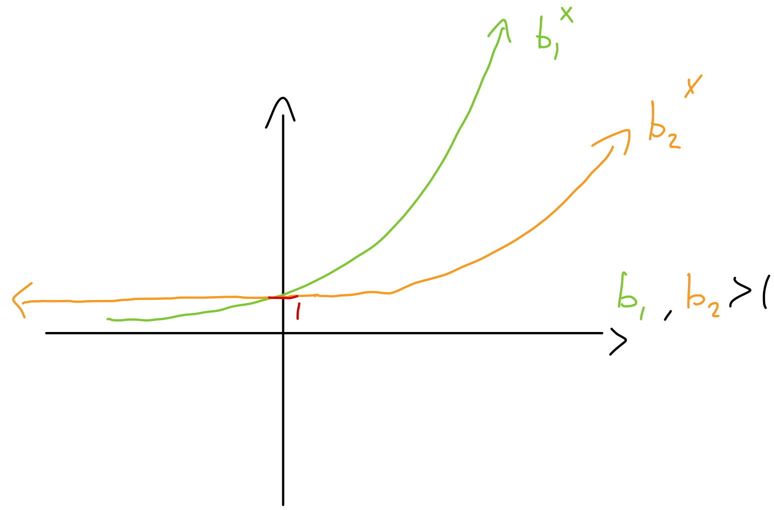
9. $f(x) = \sqrt[m]{x^n}$ $= (x^n)^{\frac{1}{m}} = x^{n\cdot\frac{1}{m}} = x^{\frac{n}{m}}$

$$f'(x) = \frac{n}{m}x^{\frac{n}{m}-1}$$

3.1.2 Exponential Functions

Exponential Functions Base e:





Let's find the slope of $g(x)=b^x$ at $x=0$.

$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$$

$$= \lim_{h\to 0} \frac{b^{0+h}-b^0}{h} = \lim_{h\to 0} \frac{b^h-1}{h}$$

We define e to be the value of b such that

$$\lim_{h\to 0}\frac{b^h-1}{h} = 1 \text{; i.e., } \boxed{\lim_{h\to 0}\frac{e^h-1}{h} = 1.} \quad (A)$$

Why is this useful?

Consider $f(x)=e^x$.

exponent property: $e^{x+h} = e^x e^h$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right)$$

$$= e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) = e^x\,(1) = e^x$$

Rule: $\frac{d}{dx}e^x = e^x$

Example. Find the derivative of $f(x)$.

1. $f(x) = \frac{e^x}{4} = \frac{1}{4}e^x$

   $f'(x) = \frac{d}{dx}\left(\frac{1}{4}e^x\right) = \frac{1}{4}\frac{d}{dx}(e^x) = \frac{1}{4}e^x$

2. just a constant number

   $f(x) = \boxed{e^3}x^4$

   $f'(x) = \frac{d}{dx}(e^3x^4) = e^3\frac{d}{dx}x^4 = e^3 \cdot 4x^{4-1} = 4e^3x^3$

3. just a constant number

   $f(x) = \boxed{e^\pi} + x^\pi + \pi e^x$

   $f'(x) = \frac{d}{dx}e^\pi + \frac{d}{dx}x^\pi + \pi\frac{d}{dx}e^x$

   $= 0 + \pi x^{\pi-1} + \pi e^x$

(remove ee^x at start)

4. $f(x) = ee^x = e^{x+1}$

   $f'(x) = \frac{d}{dx}ee^x = e\frac{d}{dx}e^x = e \cdot e^x \ldots = e^{x+1}$

You try!

5. $f(x) = 17e^x - x^{17}$

   $f'(x) = \frac{d}{dx}[17e^x - x^{17}] = 17\frac{d}{dx}e^x - \frac{d}{dx}x^{17}$

   $= 17e^x - 17x^{17-1} = 17e^x - 17x^{16}$.

Example. Find ~~all~~ all orders for derivatives of $f(x) = x^3 - 3x^2 + 2x^0$

$f'(x) = 3x^{3-1} - 3 \cdot 2x^{2-1} + 2 \cdot 0x^{0-1}$

$= 3x^2 - 6x + 0 = 3x^2 - 6x$

$\Rightarrow$ so $f'(x) = 3x^2 - 6x$

$\Rightarrow f''(x) = 3 \cdot 2 \cdot x^{2-1} - 6 \cdot 1x^{1-1}$

$= 6x^1 - 6$

so $f''(x) = 6x - 6$

$\Rightarrow f'''(x) = 6$

$\Rightarrow f^{(4)}(x) = 0$

$\Rightarrow f^{(5)}(x) = 0$

Note $f^{(n)}(x) = 0$ for $n = 4, 5, 6, \ldots$

(change wording

You try!

Example. Find all derivatives of $f(x) = 3e^x + x$

$$f'(x) = \frac{d}{dx}\left[3e^x + x\right]$$
$$= 3\frac{d}{dx}e^x + \frac{d}{dx}x$$
$$= 3e^x + \cancel{0}\ 1$$

So $\boxed{f'(x) = 3e^x + \cancel{0}\,1}$

$$\Rightarrow f''(x) = \frac{d}{dx}\left[3e^x + 1\right]$$
$$= 3\frac{d}{dx}e^x + \frac{d}{dx}1$$
$$= 3e^x + 0$$

So $\boxed{f''(x) = 3e^x}$

$\Rightarrow f'''(x) = 3e^x$

$\Rightarrow \boxed{f^{(n)}(x) = 3e^x, \quad \text{for } n = 2, 3, \ldots}$

$\Rightarrow n \geq 2 \ldots$ $\cancel{n = 2, 5}$

Not on exam!

3.2 Product and Quotient Rules

Learning Objectives: After completing this section, we should be able to

- derive and apply the Product Rule, and the Quotient Rule.

3.2.1 Product Rule

Question. *Is* $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g'(x)$*?*

Let's try $f(x) = x^2$ and $g(x) = x^3$

$f'(x) = 2x$ $g'(x) = 3x^2$

So $f'(x) \cdot g'(x) = (2x)(3x^2) = \boxed{6x^3}$

Note $f(x) \cdot g(x) = x^2 \cdot x^3 = x^{2+3} = x^5$

so $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx} x^5 = \boxed{5x^4}$

$6x^3 \neq 5x^4$

Not true!

Theorem. *The product rule states* $\frac{d}{dx}(f(x) \cdot g(x)) =$ $f(x) \cdot g'(x) + f'(x)g(x)$, if f & g are differentiable.

Example) $f(x) = x^2$ and $g(x) = x^3$

$f'(x) = 2x$ $g'(x) = 3x^2$

We know the correct answer is $\frac{d}{dx}(f(x) \cdot g(x)) = \boxed{5x^4}$

Consider $f(x) \cdot g'(x) + f'(x) \cdot g(x) = (x^2)(3x^2) + (2x)(x^3)$

$= 3x^4 + 2x^4 = \boxed{5x^4}$ ✓ Works!

Not a formal proof, just 1 example!

proof: $\frac{d}{dx}[f(x) \cdot g(x)] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$ clever 0

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) + [f(x+h)g(x) - f(x+h)g(x)] - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h)\left(\frac{g(x+h) - g(x)}{h}\right) + \lim_{h \to 0} g(x)\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \left(\lim_{h \to 0} f(x+h)\right)\left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right) + \left(\lim_{h \to 0} g(x)\right)\left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right)$$

$$= (f(x))(g'(x)) + (g(x))(f'(x))$$

So $\frac{d}{dx}[f(x)g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$, $\square$

if f & g are differentiable.

Example. Find the derivative of $h(x) = (3x^2 + 14x)(8e^x + 1)$.

First identify the two functions multiplied

$f(x) = 3x^2 + 14x$ $\qquad$ $g(x) = 8e^x + 1$

$f'(x) = 6x + 14$ $\qquad$ $g'(x) = 8e^x + 0 = 8e^x$

Note, $h(x) = f(x)\,g(x)$, so

$$h'(x) = f(x)\,g'(x) + f'(x)\,g(x)$$
$$= (3x^2+14x)(8e^x) + (6x+14)(8e^x+1)$$

No need to simplify!

You try!

Example. Find the derivative of $h(x) = (x^3 + 2)(2x^{-4} - x^{-1})$

$f(x) = x^3 + 2$ $\qquad$ $g(x) = 2x^{-4} - x^{-1}$

$f'(x) = 3x^2 + 0$ $\qquad$ $g'(x) = 2(-4)x^{-4-1} - (-1)x^{-1-1}$

$= 3x^2$ $\qquad$ $= -8x^{-5} + x^{-2}$

So
$$h'(x) = f(x)\,g'(x) + f'(x)\,g(x)$$
$$= (x^3+2)(-8x^{-5}+x^{-2}) + (3x^2)(2x^{-4} - x^{-1})$$

You try!

Example. Find the derivative of $h(x) = 4x^3e^x$

$f(x) = 4x^3$ $\qquad$ $g(x) = e^x$

$f'(x) = 4 \cdot 3x^{3-1}$ $\qquad$ $g'(x) = e^x$

$= 12x^2$

(Note $h(x) = f(x)\,g(x) = 4x^3e^x$)

$$h'(x) = f(x)\,g'(x) + f'(x)\,g(x)$$
$$= (4x^3)(e^x) + (12x^2)(e^x)$$

Example. Find the derivative of $y = e^x(x^2+1)(3x-5)$.

$f(x) = e^x$

$f'(x) = e^x$

$g(x) = (x^2+1)(3x-5)$

$g'(x)$ needs product rule!

$\bar{f}(x) = x^2+1$ $\bar{g}(x) = 3x-5$

$\bar{f}'(x) = 2x+0$ $\bar{g}'(x) = 3$

$= 2x$

so $g'(x) = \bar{f}(x)\cdot\bar{g}'(x) + \bar{f}'(x)\bar{g}(x)$

$= (x^2+1)(3) + (2x)(3x-5)$

$\Rightarrow y' = f(x)\, g'(x) + f'(x)\, g(x)$

$= (e^x)\left[(x^2+1)(3) + (2x)(3x-5)\right] + (e^x)\left[(x^2+1)(3x-5)\right]$

You try!

Example. Evaluate $\frac{d}{dx}(xe^{2x})$.

Note $e^{2x} = e^{x+x} = e^x e^x$,

so $\frac{d}{dx}(xe^{2x}) = \frac{d}{dx}(x e^x e^x)$

$f(x) = xe^x$

$g(x) = e^x$

$g'(x) = e^x$

$f'(x)$ requires product rule

$u(x) = x$ $v(x) = e^x$

$u'(x) = 1$ $v'(x) = e^x$

$f'(x) = u(x)\cdot v'(x) + u'(x)\cdot v(x)$

$= (x)(e^x) + (1)(e^x)$

$\frac{d}{dx}(xe^{2x}) = f(x)\, g'(x) + f'(x)\, g(x)$

$= (xe^x)(e^x) + \left[(x)(e^x) + (1)(e^x)\right](e^x)$

stop here

$= xe^{2x} + xe^x e^x + e^x e^x$

$= xe^{2x} + xe^{2x} + e^{2x} = 2xe^{2x} + e^{2x}$

3.2.2 Quotient Rule

Theorem. *The quotient rule states* $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$, where f & g are differentiable and $g(x) \neq 0$.

$$Q = \frac{\text{high}}{\text{low}} \quad \text{then} \quad Q' = \frac{(\text{Low})(\text{D-High}) - (\text{High})(\text{D-Low})}{(\text{be Low})^2}$$

"Low D-High, High D-Low, 2 beLow."
(the rhymes D-Low and beLow should be next eachother)

Example. Find the derivative of $h(x) = \frac{x^4-8x^2}{x-1}$.

$f(x) = x^4 - 8x^2$ $\qquad$ $g(x) = x - 1$

$f'(x) = 4x^3 - 8\cdot 2x^1 = 4x^3 - 16x$ $\qquad$ $g'(x) = 1$

$$h'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{(g(x))^2}$$

$$= \frac{(x-1)(4x^3-16x) - (x^4-8x^2)(1)}{(x-1)^2}.$$

You try!

Example. Find the derivative of $y = \frac{e^x - 2x}{1 - xe^x}$.

$f(x) = e^x - 2x$

$f'(x) = e^x - 2$

$g(x) = 1 - xe^x$

$g'(x) = 0 - \underbrace{\frac{d}{dx}(xe^x)}_{\text{product rule}}$

$u(x) = x \qquad v(x) = e^x$

$u'(x) = 1 \qquad v'(x) = e^x$

$g'(x) = 0 - [u(x)\cdot v'(x) + u'(x)v(x)]$

$= -[x\cdot e^x + 1\cdot e^x]$

$$y' = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{(g(x))^2}$$

$$= \frac{(1-xe^x)(e^x-2) - (e^x-2x)\left(-[xe^x + 1\cdot e^x]\right)}{(1-xe^x)^2}$$

Example. Find the derivative of $y = \frac{3x - e^x}{2}$.

$$y = \frac{3x}{2} - \frac{e^x}{2} = \frac{3}{2}x - \frac{1}{2}e^x$$

$$y' = \frac{3}{2} \cdot 1 - \frac{1}{2}e^x$$

If the denominator is a constant, then the quotient rule is never necessary

Example. Find the derivative of $y = \frac{2x + e^x}{x}$.

$$y = \frac{2x}{x} + \frac{e^x}{x} = 2 + \frac{e^x}{x}$$

$$y' = 0 + \frac{d}{dx}\left(\frac{e^x}{x}\right)$$

$f(x) = e^x$ $\quad g(x) = x$

$f'(x) = e^x$ $\quad g'(x) = 1$

$$y' = 0 + \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = 0 + \frac{(x)(e^x) - (e^x)(1)}{(x)^2}$$

Use quotient rule immediately

$f(x) = 2x + e^x$ $\quad g(x) = x$

$\vdots$

you will get same answer (after simplification)

Example. Find the derivative of $y = \frac{x^\pi - \sqrt{x}}{x^3}$.

$$y = \frac{x^\pi}{x^3} - \frac{\sqrt{x}}{x^3} = \frac{x^\pi}{x^3} - \frac{x^{\frac{1}{2}}}{x^3} = x^{\pi - 3} - x^{\frac{1}{2} - 3}$$

$$\Rightarrow y = x^{\pi - 3} - x^{-5/2}$$

$$y' = (\pi - 3)x^{\pi - 3 - 1} - \left(-\frac{5}{2}\right)x^{-\frac{5}{2} - 1}$$

If the denominator depends on x, then you may be able to simply to avoid quotient rule, but it may not always work

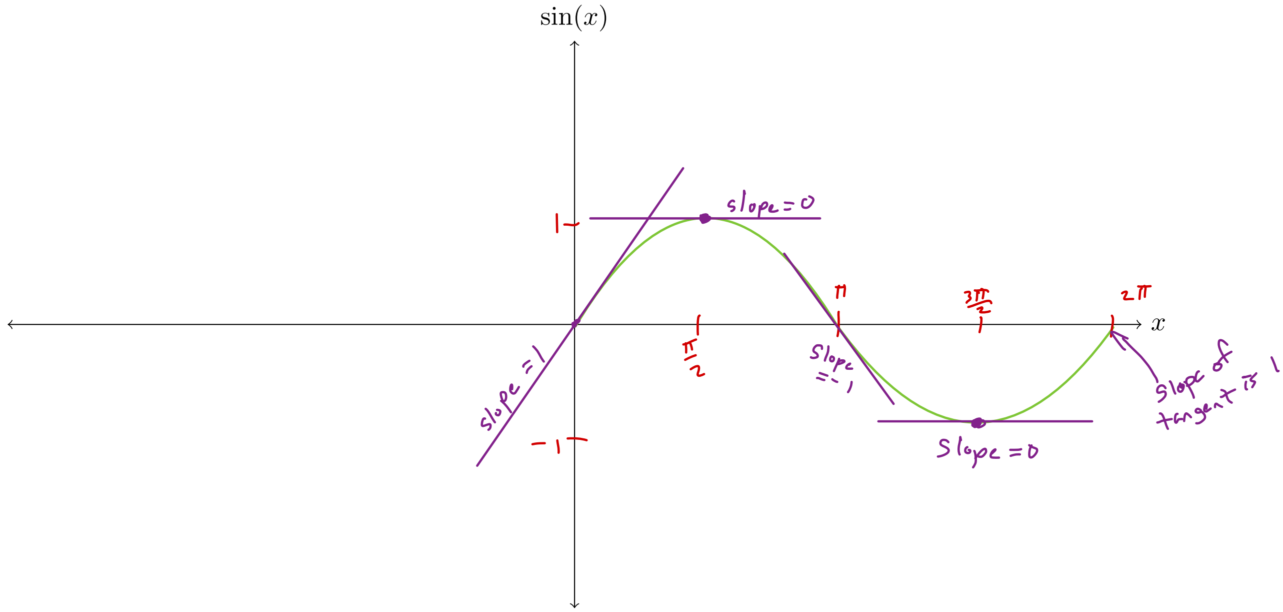
Not on Exam 1

3.3 Derivatives of Trigonometric Functions

Learning Objectives: After completing this section, we should be able to

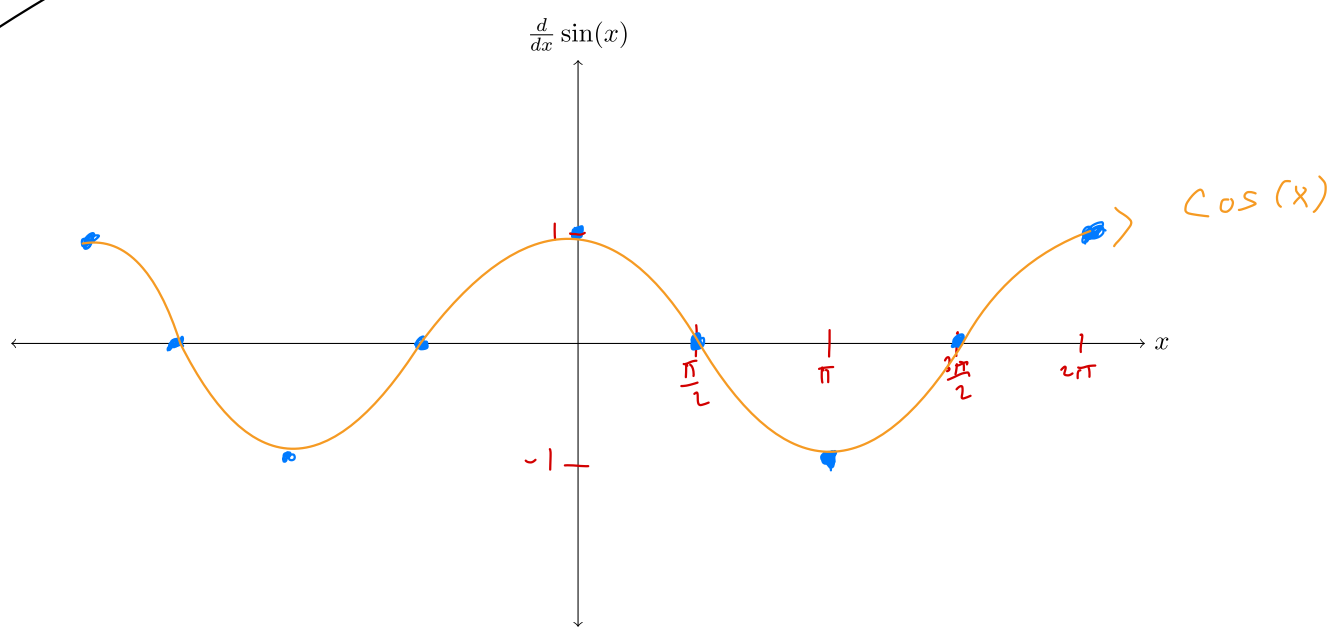
- derive and apply derivatives for trigonometric functions.

We are going to focus on derivatives of trigonometry functions.

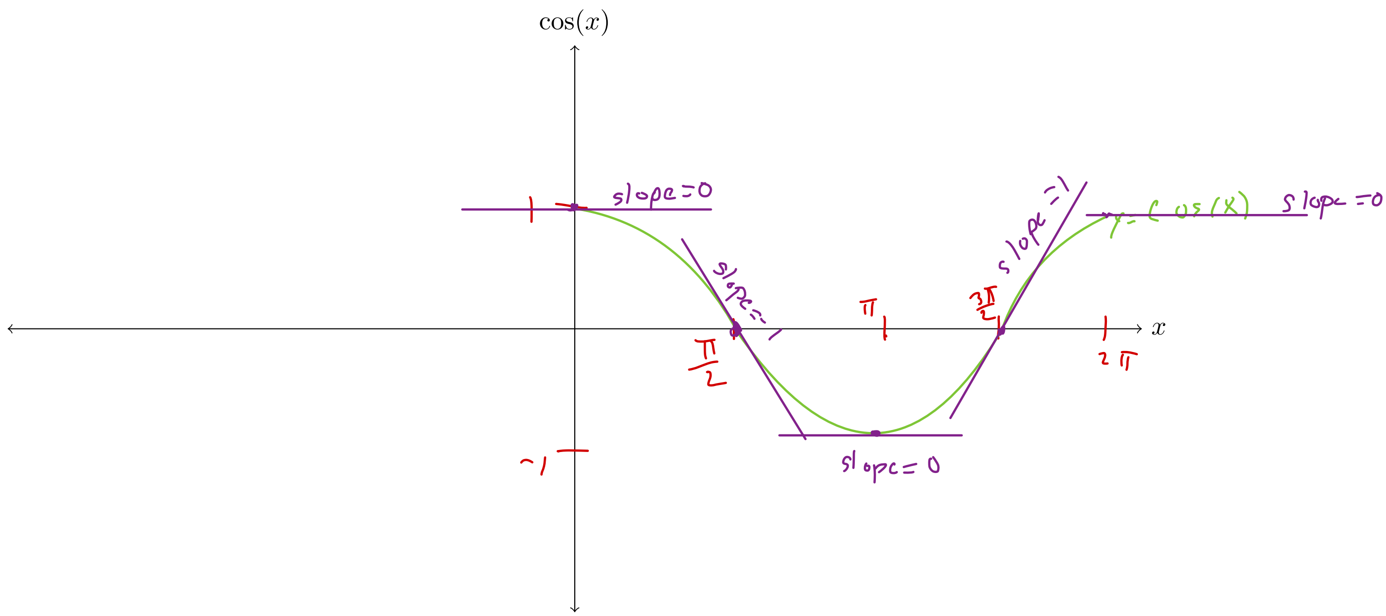


Let's graph it's derivative

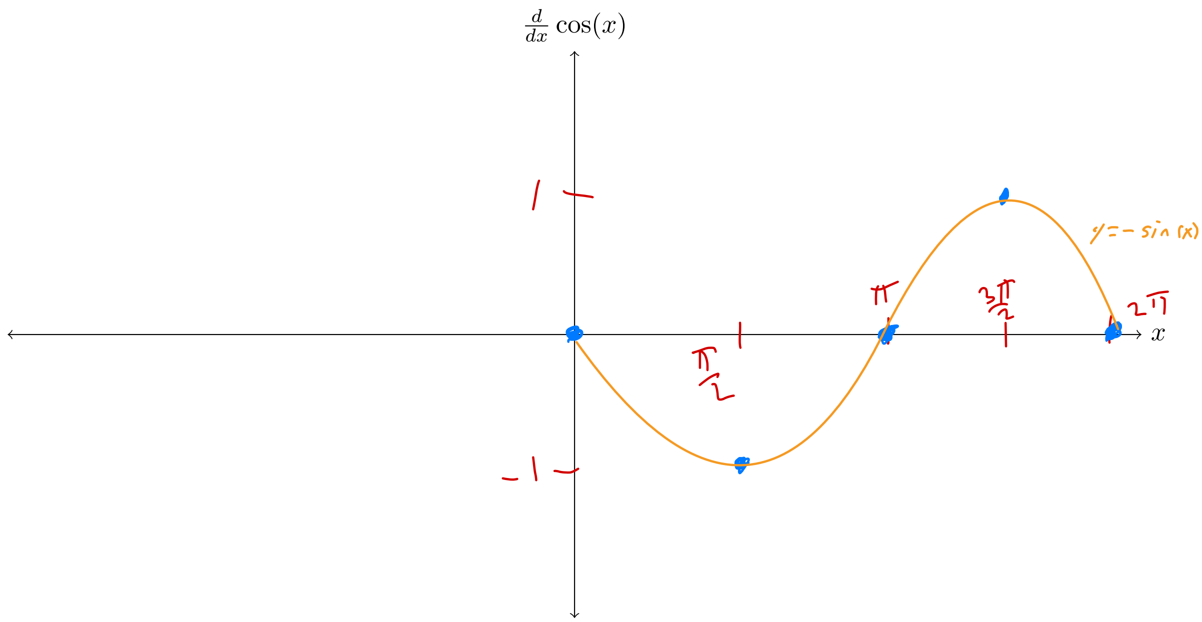
$e^{\pi i} = -1$



Rule: $\frac{d}{dx}\sin(x) = \cos(x)$



Let's graph it's derivative



Rule: $\frac{d}{dx}\cos(x) = -\sin(x)$

Example. Find the derivative of $f(x) = \tan(x)$.

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

Quotient rule:

$U(x) = \sin(x)$ $\quad$ $v(x) = \cos(x)$

$u'(x) = \cos(x)$ $\quad$ $v'(x) = -\sin(x)$

$$f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{(v(x))^2}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

$$= (\sec(x))^2$$

$\cos^2(x) + \sin^2(x) = 1$

$\sec(x) = \frac{1}{\cos(x)}$

Using the quotient rule, we can find the derivatives of $\csc(x)$, $\cot(x)$, and $\sec(x)$. Simplify!

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\frac{\sin(x)}{\cos(x)} = \tan(x)$	$\sec^2(x)$
$\frac{1}{\sin(x)} = \csc(x)$	$-\csc(x)\cdot\cot(x)$
$\frac{1}{\cos(x)} = \sec(x)$	$\sec(x)\cdot\tan(x)$
$\frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$	$-\csc^2(x)$

(Note, derivatives of trig fnctns starting with c get a negative sign)

You try!

Example. Find the derivative of $f(x)$ for the following problems.

1. $f(x) = e^x \sin(x)$.

$$\text{left} = e^x \qquad \text{right} = \sin(x)$$
$$\text{left}' = e^x \qquad \text{right}' = \cos(x)$$

$$f'(x) = (\text{left})(\text{right}') + (\text{left}')(\text{right})$$
$$= e^x \cdot \cos(x) + e^x \cdot \sin(x)$$

2. $f(x) = \frac{x\tan(x)}{1+\cos(x)}$.

$$\text{high} = x \cdot \tan(x) \qquad \text{low} = 1 + \cos(x)$$
$$\text{left} = x \qquad \text{right} = \tan(x) \qquad \text{low}' = 0 + -\sin(x) = -\sin(x)$$
$$\text{left}' = 1 \qquad \text{right}' = \sec^2(x)$$
$$\text{high}' = (\text{left})(\text{right}') + (\text{left}')(\text{right})$$
$$= x \cdot \sec^2(x) + 1 \cdot \tan(x)$$
$$f'(x) = \frac{(\text{low})(\text{high}') - (\text{high})(\text{low}')}{(\text{low})^2} = \frac{(1+\cos(x))[x \cdot \sec^2(x) + \tan(x)] - (x \cdot \tan(x))(-\sin(x))}{(1+\cos(x))^2}$$

3. $f(x) = \sec(x)\tan(x)$.

$$f'(x) = (\sec(x)) \cdot (\sec^2(x)) + (\sec(x) \cdot \tan(x))(\tan(x))$$

4. $f(x) = \cot(x)\cos(x)$.

$$f'(x) = (\cot(x))(-\sin(x)) + (-\csc^2(x))(\cos(x))$$

3.4 Chain Rule

Learning Objectives: After completing this section, we should be able to

- apply the chain rule to obtain the derivative of a composite function.
- apply the chain rule to obtain the derivative of a power function and an exponential function.

Example. Let's try to find the derivative of $y = (3x^2 + x)^2$.

can't use the power rule right away, as $3x^2+x$ is not just x

$y = (3x^2+x)(3x^2+x)$

left $= 3x^2+x$ right $= 3x^2+x$

left$' = 6x+1$ right$' = 6x+1$

$y' = (\text{left})(\text{right}') + (\text{left}')(\text{right})$

$= (3x^2+x)(6x+1) + (6x+1)(3x^2+x)$

$= 2(3x^2+x)(6x+1)$

What if we tried the power rule?

$y = (3x^2+x)^2 \Rightarrow y' = 2(3x^2+x)^{2-1}\,\boxed{}$

$= 2(3x^2+x)\,\boxed{}$

What's missing $\boxed{}$ in y'? $\boxed{} = 6x+1$, which is the derivative of $3x^2+x$.

Theorem. *The chain rule states* $\frac{d}{dx} f(g(x)) = \left[f'(g(x))\right] \cdot \left[g'(x)\right]$

Example. Again, consider the function $y = (3x^2 + x)^2$.

Let $g(x) = 3x^2+x$ $f(x) = x^2$ $\Rightarrow f(☺) = (☺)^2$

$g'(x) = 6x+1$ $f'(x) = 2x$ $\Rightarrow f'(☺) = 2(☺)$

Note $f(g(x)) = f(3x^2+x) = (3x^2+x)^2 = y$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(3x^2+x) \cdot (6x+1)$

$= 2(3x^2+x) \cdot (6x+1)$

Example. $y = (4x^2 + x^{-2})^4$.

$f(u) = u^4$

$f'(u) = 4u^3$

$g(x) = 4x^2 + x^{-2}$

$g'(x) = 8x + (-2)x^{-2-1}$

$= 8x - 2x^{-3}$

$f(g(x)) = f(4x^2 + x^{-2}) = (4x^2 + x^{-2})^4 = y$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(4x^2 + x^{-2}) \cdot (8x - 2x^{-3})$

$= 4(4x^2 + x^{-2})^3 \cdot (8x - 2x^{-3})$

Originally missed 2

Example. $y = \cos(x^2 - 1)$.

$f(x) = x^2 - 1$

$f'(x) = 2x$

$g(x) = \cos(x)$

$g'(x) = -\sin(x)$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(\cos(x)) \cdot (-\sin(x))$

$= 2(\cos(x)) \cdot (-\sin(x))$

$f(u) = \cos(u)$

$f'(u) = -\sin(u)$

$g(x) = x^2 - 1$

$g'(x) = 2x$

$f(g(x)) = f(x^2 - 1) = \cos(x^2 - 1) = y$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(x^2 - 1) \cdot (2x)$

$= -\sin(x^2 - 1) \cdot (2x)$

You try!

Example. $y = \tan(5x^2 + 2x)$.

$f(u) = \tan(u)$

$f'(u) = \sec^2(u)$

$g(x) = 5x^2 + 2x$

$g'(x) = 10x + 2$

$y = f(g(x)) = f(5x^2 + 2x) = \tan(5x^2 + 2x)$

$\Rightarrow \quad y' = f'(g(x)) \cdot g'(x)$

$= f'(5x^2 + 2x) \cdot (10x + 2)$

$= \sec^2(5x^2 + 2x) \cdot (10x + 2)$

We can combine rules.

Example. $y = \cos^3(x^2 - 1)$. $= \left(\cos(x^2-1)\right)^3$

$f(u) = u^3$

$f'(u) = 3u^2$

$g(x) = \cos(x^2-1)$

$g'(x) =$ (chain rule problem from 2 examples ago)

$= -\sin(x^2-1)(2x)$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(\cos(x^2-1)) \cdot (-\sin(x^2-1) \cdot (2x))$

$= 3(\cos(x^2-1))^2 \cdot (-\sin(x^2-1)) \cdot (2x)$

You try!

Example. $y = \sqrt{\sin(3x)}$. $\Rightarrow$ $y = \left(\sin(3x)\right)^{\frac{1}{2}}$

$f(u) = u^{1/2}$

$f'(u) = \frac{1}{2} u^{-\frac{1}{2}}$

$g(x) = \sin(3x)$

$g'(x) = \ldots$ chain rule!

$\bar{f}(u) = \sin(u)$

$\bar{f}'(u) = \cos(u)$

$\bar{g}(x) = 3x$

$\bar{g}'(x) = 3$

$g'(x) = \bar{f}'(\bar{g}(x)) \cdot \bar{g}'(x)$

$= \bar{f}'(3x) \cdot 3 = \cos(3x) \cdot 3$

$y' = f'(g(x)) \cdot g'(x) = f'(\sin(3x)) \cdot [\cos(3x) \cdot 3]$

$= \frac{1}{2}(\sin(3x))^{-\frac{1}{2}} \cdot [\cos(3x) \cdot 3]$

You try!

Example. $y = \left(\dfrac{\sin(x)}{1+\cos(x)}\right)^5$.

$f(u) = u^5$

$f'(u) = 5u^4$

$g(x) = \dfrac{\sin(x)}{1+\cos(x)}$

$g'(x) = \ldots$ quotient rule!

low $= 1+\cos(x)$

low$' = 0 - \sin(x)$

high $= \sin(x)$

high$' = \cos(x)$

$$g'(x) = \frac{(1+\cos(x)) \cdot \cos(x) - \sin(x)(0 - \sin(x))}{(1+\cos(x))^2} = \frac{(\text{low})(\text{high}') - (\text{high})(\text{low}')}{(\text{low})^2}$$

So $y' = f'(g(x)) \cdot g'(x)$

$= f'\left(\dfrac{\sin(x)}{1+\cos(x)}\right) \cdot g'(x)$

$$= 5\left(\frac{\sin(x)}{1+\cos(x)}\right)^4 \cdot \frac{(1+\cos(x)) \cdot \cos(x) - \sin(x)(0 - \sin(x))}{(1+\cos(x))^2}$$

Recall we have the derivative of $y = e^{2x}$ is $y' = 2e^{2x}$. Let's think of it with the chain rule.

$f(u) = e^u \qquad g(x) = 2x$
$f'(u) = e^u \qquad g'(x) = 2$

$y' = f'(g(x)) \cdot g'(x) = f'(2x) \cdot 2 = e^{2x} \cdot 2$

constant chain shortcut: If $y = e^{c \cdot x}$, then $y' = e^{c \cdot x} \cdot c$, for any constant c

what about $y = 5^x$? $\Rightarrow$ $\ln(y) = \ln(5^x) = x \cdot \ln(5)$

$\Rightarrow e^{\ln(y)} = e^{x \cdot \ln(5)} \Rightarrow y = e^{\ln(5) \cdot x} = 5^x$

$\Rightarrow y' = e^{\ln(5) \cdot x} \cdot \ln(5) = 5^x \cdot \ln(5)$

General Exponential Derivative: If $y = b^x$, then $y' = b^x \cdot \ln(b)$, ~~if~~ for $b > 0$.

You try!

Example. $y = e^{\left(4x^2 - 5x + 1\right)}$.

$f(u) = e^u \qquad g(x) = 4x^2 - 5x + 1$
$f'(u) = e^u \qquad g'(x) = 8x - 5$

$y' = f'(g(x)) \cdot g'(x)$
$= f'(4x^2 - 5x + 1) \cdot (8x - 5)$
$= \left(e^{4x^2 - 5x + 1}\right)(8x - 5)$

General Exponential chain rule: If $y = e^{g(x)}$, then $y' = e^{g(x)} \cdot g'(x)$.

You try!

Example. $y = 2e^{\left(3\cos^2(x^4)\right)}$.

$y' = $ (general exponential chain rule) $= 2e^{3\cos^2(x^4)} \left[\frac{d}{dx} 3\cos^2(x^4)\right]$

$\frac{d}{dx} 3\cos^2(x^4)$ requires chain rule.

$3\cos^2(x^4)$
$= 3\left(\cos(x^4)\right)^2$

$f(u) = 3 \cdot u^2 \qquad g(x) = \cos(x^4)$
$f'(u) = 6 \cdot u \qquad g'(x) = \ldots$ chain rule!

outside $= \cos(u)$
outside$' = -\sin(u)$

inside $= x^4$
inside$' = 4x^3$

$g'(x) = $ outside$'$(inside) $\cdot$ (inside$'$)
$= -\sin(x^4) \cdot 4x^3$

$\frac{d}{dx} 3\cos^2(x^4) = f'(g(x)) \cdot g'(x)$
$= 6(\cos(x^4)) \cdot [-\sin(x^4) \cdot 4x^3]$

$\Rightarrow y' = 2e^{3\cos^2(x^4)} \left(6(\cos(x^4)) \cdot [-\sin(x^4) \cdot 4x^3]\right)$

Forgot the quotient rule? No problem!

Product rule: $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + f'(x)\cdot g(x)$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left[f(x)\cdot (g(x))^{-1}\right]$$

left $= f(x)$ right $= (g(x))^{-1}$

left$' = f'(x)$ right$' = (-1)(g(x))^{-2}\cdot g'(x)$

chain rule where outside $= u^{-1}$
inside $= g(x)$

$$\frac{d}{dx}\left[f(x)\cdot(g(x))^{-1}\right] = (\text{left})(\text{right}') + (\text{left}')(\text{right})$$
$$= f(x)(-1)(g(x))^{-2}\cdot g'(x) + f'(x)(g(x))^{-1}$$
$$= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}\left(\frac{g(x)}{g(x)}\right) = -\frac{f(x)g'(x)}{(g(x))^2} + \frac{f'(x)g(x)}{(g(x))^2}$$
$$= \frac{f'(x)g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

When do we need the quotient rule?

1. $y = \dfrac{x^4+e^x}{5}. = \dfrac{x^4}{5} + \dfrac{e^x}{5} = \dfrac{1}{5}x^4 + \dfrac{1}{5}e^x$

$$y' = \frac{1}{5}4x^3 + \frac{1}{5}e^x$$

(constant in denominator, we do not need quotient rule)

2. $y = \dfrac{5}{x^4+e^x}. = 5(x^4+e^x)^{-1}$ (chain rule)

outside $= 5u^{-1}$ inside $= x^4+e^x$

outside$' = 5(-1)u^{-2}$ inside$' = 4x^3+e^x$

(constant in numerator can be written as a chain rule problem)

$$y' = \text{outside}'(\text{inside})\cdot(\text{inside}')$$
$$= 5(-1)(x^4+e^x)^{-2}\cdot(4x^3+e^x)$$

3. $y = \dfrac{x^4+e^x}{x-1}.$

(If both numerator and denominator are functions of x and there are no immediate cancellations, use quotient rule)

3.6 Derivatives of Logarithms and Inverse Trigonometric Functions

Learning Objectives: After completing this section, we should be able to

- define the general logarithmic functions
- derive the derivative of logarithmic functions.
- derive the derivatives of all inverse trigonometric functions.

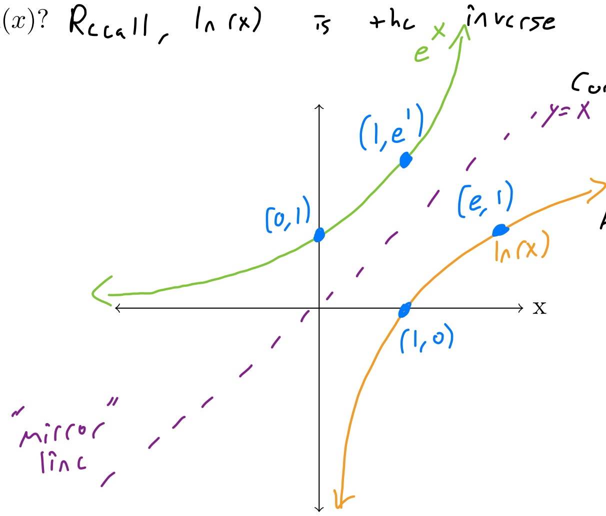
3.6.1 Logarithmic Functions

Key properties of logarithms:

1. $\log(b^x) =$ $x \cdot \log(b)$
2. $\log(x \cdot y) =$ $\log(x) + \log(y)$
3. $\log\left(\frac{x}{y}\right) =$ $\log(x) - \log(y)$
4. $\log(x + y) =$ $\log(x+y)$

   No possible simplification

What is $y = \log_e(x) = \ln(x)$? Recall, $\ln(x)$ is the inverse fnctn of e^x; i.e.,
(Naturally logarithm)



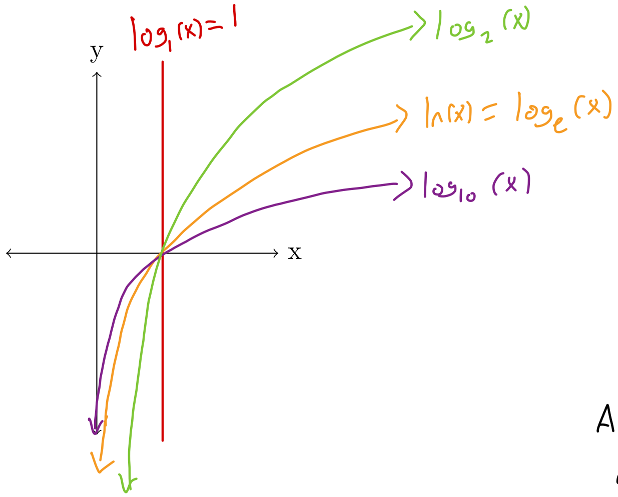
Consider $\ln(e^x) = x \cdot \ln(e)$
$= x \cdot 1$
$= x$

Also $e^{\ln(x)} = x$

Other Bases:

$\log_b(x)$ is the inverse of b^x; e.g.,

- $\log_{10}(10^x) = x$
- $2^{\log_2(x)} = x$



In general, $\log_b(x)$ goes through the points $(1, 0)$ and $(b, 1)$

Also, $\log_b(x)$ has a vertical asymptote at $x=0$ if $b>1$.

Logarithms and exponentials have an inverse relationship.

$$\ln(e^x) = x \quad \text{or} \quad e^{\ln(x)} = x$$

$$\log_b(b^x) = x \quad \text{or} \quad b^{\log_b(x)} = x$$

Question. *How do we take the derivative of a logarithm?*

Let $y = \ln(x)$. This implies $x = e^y$, as $e^{(y)} = e^{(\ln(x))} \Rightarrow e^y = x$

To find the derivative of $y=\ln(x)$, it is equivalent to find $\frac{dy}{dx}$

So, we will implicitly differentiate $x = e^y$ to find $\frac{dy}{dx}$.

1) $$\frac{d}{dx} x = \frac{d}{dx} e^y$$

$$\Rightarrow 1 = e^y \cdot \frac{dy}{dx}$$

2) $$\frac{1}{e^y} = \frac{e^y}{e^y} \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{e^y} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}, \quad \text{as} \quad x = e^y$$

$$\Rightarrow \frac{d}{dx} \ln(x) = \frac{1}{x}$$

- Note, if $x<0$, then $y = \ln(x)$ is undefined, but $y = \ln(-x)$ is defined, e.g., if $x=-4$, then $y = \ln(-4)$ is undefined, but $y = \ln(-(-4)) = \ln(4)$ is defined

Consider $y = \ln(-x) \Rightarrow e^y = -x$

$$\frac{d}{dx} e^y = \frac{d}{dx} -x$$

$$\Rightarrow e^y \frac{dy}{dx} = -1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{e^y} = \frac{-1}{-x} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx} \ln(|x|) = \frac{1}{x}, \quad \text{if} \quad x \neq 0$$

- Other bases: $y = \log_b(x) \Rightarrow b^y = x$

$$\Rightarrow \frac{d}{dx} b^y = \frac{d}{dx} x$$

(Recall, $\frac{d}{dx} b^x = b^x \cdot \ln(b)$)

$$\Rightarrow b^y \cdot \ln(b) \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b^y \ln(b)} = \frac{1}{x \cdot \ln(b)}$$

$$\Rightarrow \frac{d}{dx} \log_b(x) = \frac{1}{x \cdot \ln(b)}$$

Let's practice! Recall $\frac{d}{dx} b^x = b^x \cdot \ln(b)$

1. $y = 2^x$

$y' = 2^x \cdot \ln(2)$

2. $y = \pi^x$

$y' = \pi^x \cdot \ln(\pi)$

3. $y = 3^{x^2+2x+1}$

$f(u) = 3^u$ $\quad g(x) = x^2+2x+1$

$f'(u) = 3^u \cdot \ln(3)$ $\quad g'(x) = 2x+2$

so $y' = f'(g(x)) \cdot g'(x)$

$= f'(x^2+2x+1) \cdot (2x+2)$

$= 3^{x^2+2x+1} \cdot \ln(3) \cdot (2x+2)$

4. $y = 8 \cdot 3^x$

$y' = 8 \cdot \frac{d}{dx} 3^x = 8 \cdot 3^x \cdot \ln(3)$

5. $y = 4^{-x}$ chain rule

$y' = \underbrace{4^{-x} \cdot \ln(4)}_{\text{deriv. of } 4^u} \cdot \underbrace{(-1)}_{\text{deriv. of } -x}$

or $y = 4^{-x} = (4^{-1})^x = \left(\frac{1}{4}\right)^x$

$y' = \left(\frac{1}{4}\right)^x \cdot \ln\left(\frac{1}{4}\right)$

6. $y = 4\ln(3x)$

$f(u) = 4 \cdot \ln(u)$ $\quad g(x) = 3x$

$f'(u) = 4 \cdot \frac{1}{u}$ $\quad g'(x) = 3$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(3x) \cdot 3$

$= 4 \cdot \frac{1}{3x} \cdot 3$

7. $y = \ln(12x^5) + 12x^5$

$f(u) = \ln(u)$ $\quad g(x) = 12x^5$

$f'(u) = \frac{1}{u}$ $\quad g'(x) = 12 \cdot 5 \cdot x^4$

$\Rightarrow y' = f'(g(x)) \cdot g'(x) + \frac{d}{dx} 12x^5$

$= f'(12x^5) \cdot 12 \cdot 5 \cdot x^4 + 12 \cdot 5 \cdot x^4$

$= \frac{1}{12x^5} \cdot 12 \cdot 5 \cdot x^4 + 12 \cdot 5 \cdot x^4$

8. $y = \ln(\tan(x))$

$f(u) = \ln(u)$ $\quad g(x) = \tan(x)$

$f'(u) = \frac{1}{u}$ $\quad g'(x) = \sec^2(x)$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(\tan(x)) \cdot \sec^2(x)$

$= \frac{1}{\tan(x)} \cdot \sec^2(x)$

9. $y = \log_5(3x)$

$f(u) = \log_5(u)$ $\quad g(x) = 3x$

$f'(u) = \frac{1}{u \cdot \ln(5)}$ $\quad g'(x) = 3$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(3x) \cdot 3$

$= \frac{1}{3x \cdot \ln(5)} \cdot 3$

10. $y = \log_b(\tan(x))$

$f(u) = \log_b(u)$ $\quad g(x) = \tan(x)$

$f'(u) = \frac{1}{u \cdot \ln(b)}$ $\quad g'(x) = \sec^2(x)$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(\tan(x)) \cdot \sec^2(x)$

$= \frac{1}{\tan(x) \cdot \ln(b)} \cdot \sec^2(x)$

11. $y = \log_2(e^x)$

$f(u) = \log_2(u)$ $\quad g(x) = e^x$

$f'(u) = \frac{1}{u \cdot \ln(2)}$ $\quad g'(x) = e^x$

$\Rightarrow y' = f'(g(x)) \cdot g'(x)$

$= f'(e^x) \cdot e^x$

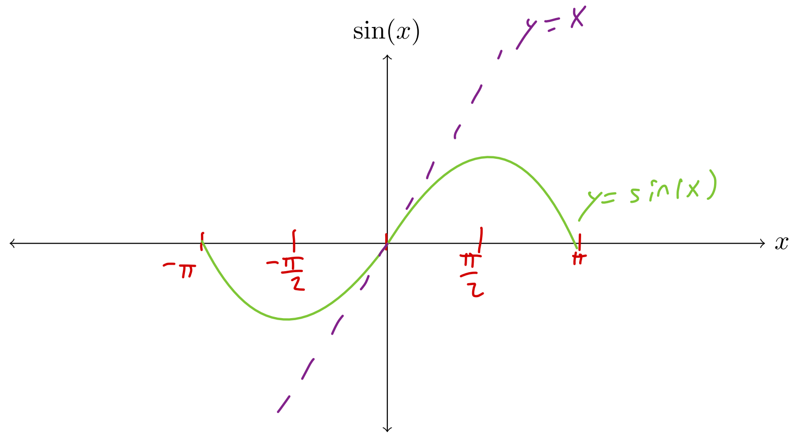
$= \frac{1}{e^x \cdot \ln(2)} \cdot e^x = \cdots = \frac{1}{\ln(2)}$

3.6.2 Inverse Trigonometric Functions

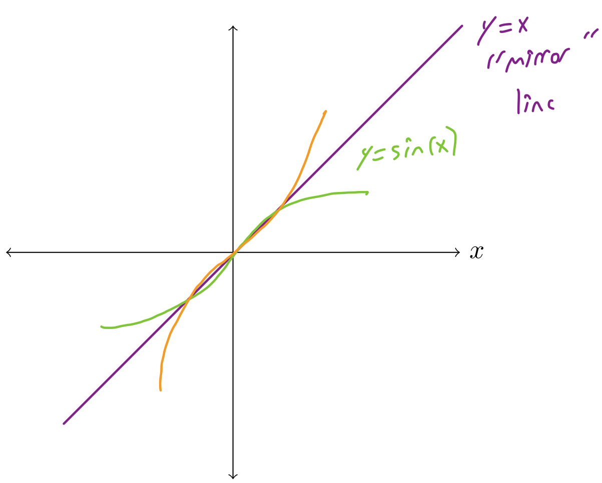
Consider $y = \sin^{-1}(x) =$ $\arcsin(x)$ is the inverse of $\sin(x)$, i.e., $\arcsin(\sin(x)) = x$ or $\sin(\arcsin(x)) = x$

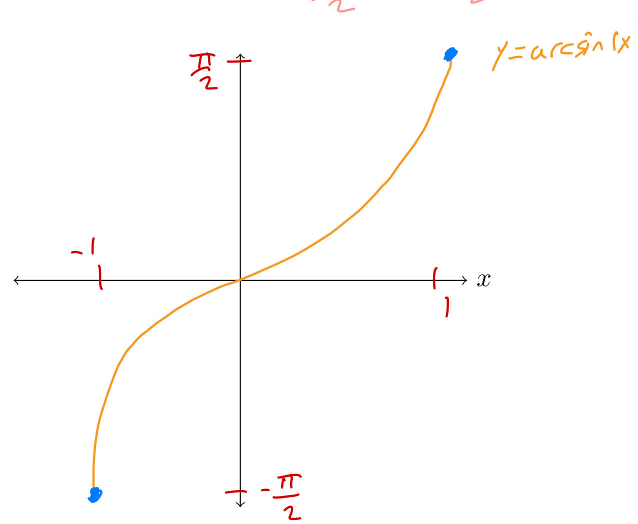
Note $y = \sin^{-1}(x) \neq \frac{1}{\sin(x)}$

Let's sketch a graph of $\sin(x)$ (over $[-\pi, \pi]$)



To find inverse functions, we "replace" x and y. Graphically, this is equivalent to mirroring the functions' graph over the line y=x. If we mirrored all of sin(x), then we would fail the vertical line test. So, we restrict the domain of sin(x) to $\cancel{-\pi}$ $-\frac{\pi}{2}$ to $\cancel{\pi}$ $\frac{\pi}{2}$ when considering inverse sin(x)





domain of $\sin(x)$ (restricted) $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] =$ range of $\arcsin(x)$

range of $\sin(x) = [-1, 1] =$ domain of $\arcsin(x)$

Example. Compute $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$.

Intentionally omitted

Example. Compute $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right) =$

Intentionally omitted

Example. Compute $\arccos\left(\frac{\sqrt{3}}{2}\right) =$

Intentionally omitted

Example. Compute $\arctan\left(-\frac{1}{\sqrt{3}}\right) =$

Intentionally omitted

How do we find derivatives of inverse trig?

Example. Let $y = \arctan(x) = \tan^{-1}(x)$

this implies $x = \tan(y)$

The derivative of $\arctan(x)$ is $\frac{dy}{dx}$

1) $\frac{d}{dx} x = \frac{d}{dx} \tan(y)$

$1 = \sec^2(y) \cdot \frac{dy}{dx}$

2) $\Rightarrow \frac{dy}{dx} = \frac{1}{\underbrace{\sec^2(y)}}$

want a formula that does not depend on y

Note, $x = \tan(y) = \frac{(opp)}{(adj)}$

$\tan(y) = \frac{x}{1} \quad \frac{(opp)}{(adj)}$

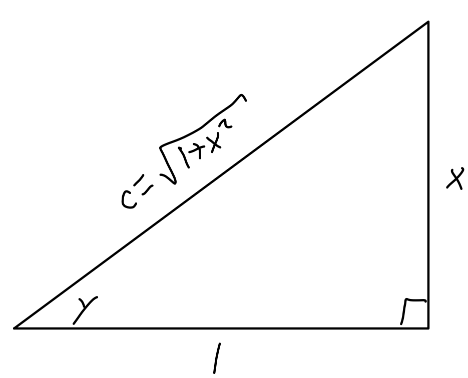
Recall, $\sec(y) = \frac{1}{\cos(y)} = \frac{1}{\frac{(adj)}{(hyp)}}$

$= \frac{(hyp)}{(adj)} = \frac{\sqrt{1+x^2}}{1}$

$\Rightarrow \sec(y) = \sqrt{1+x^2}$

$\Rightarrow \sec^2(y) = 1 + x^2$

$\boxed{\Rightarrow \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}}$



$a^2 + b^2 = c^2$

$1^2 + x^2 = c^2$

$\Rightarrow c = \sqrt{1^2 + x^2} = \sqrt{1+x^2}$

Example. Let $y = 10\arctan(4x^2)$. Find y'.

$$\text{outside} = 10\arctan(u) \qquad \text{inside} = 4x^2$$
$$\text{outside}' = 10\frac{1}{1+u^2} \qquad \text{inside}' = 8x$$

$$y' = \text{outside}'(\text{inside}) \cdot \text{inside}'$$
$$= \text{outside}'(4x^2) \cdot 8x$$
$$= 10\frac{1}{1+(4x^2)^2} \cdot 8x$$

You try!

Example. Find a formula for $\frac{d}{dx}\arcsin(x)$.

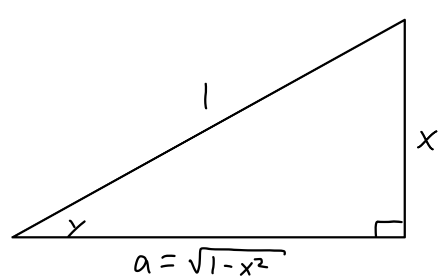
$$y = \arcsin(x) \Rightarrow x = \sin(y)$$

$$\sin(y) = x = \frac{x}{1}$$

$$\frac{d}{dx}x = \frac{d}{dx}\sin(y)$$
$$1 = \cos(y) \cdot \frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Note, $\sin(y) = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$

So $\cos(y) = \frac{(\text{adj})}{(\text{hyp})}$
$$= \frac{\sqrt{1-x^2}}{1}$$
$$= \sqrt{1-x^2}$$



$$a^2 + b^2 = c^2$$
$$a^2 + x^2 = 1^2 = 1$$
$$\Rightarrow a^2 = 1 - x^2$$
$$\Rightarrow a = \sqrt{1-x^2}$$

So $\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$

$$\boxed{\Rightarrow \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$$

You try!

Example. Find a formula for $\dfrac{d}{dx}\arccos(x)$.

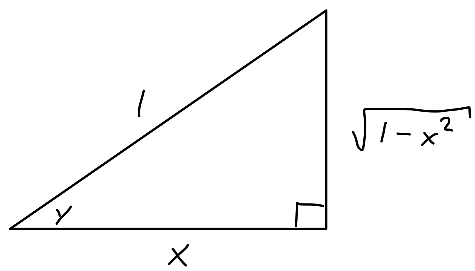
$$y = \arccos(x) \implies x = \cos(y)$$

$$\frac{d}{dx}x = \frac{d}{dx}\cos(y)$$

$$\implies 1 = -\sin(y)\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = -\frac{1}{\sin(y)}$$

Recall $x = \cos(y) = \dfrac{(\text{adj})}{(\text{hyp})} \implies \cos(y) = \dfrac{x}{1} \quad \dfrac{(\text{adj})}{(\text{hyp})}$



So $\sin(y) = \dfrac{(\text{opp})}{(\text{hyp})} = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$$\implies \frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\implies \frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}}$$

3.6.3 Summary

- $\frac{d}{dx}e^{f(x)} =$ $e^{f(x)} \cdot f'(x)$
- $\frac{d}{dx}b^{f(x)} =$ $b^{f(x)} \cdot \ln(b) \cdot f'(x)$

Power rule does not apply

- $\frac{d}{dx}\ln(f(x)) =$ $\frac{1}{f(x)} \cdot f'(x),$

(as, $\frac{d}{dx}\ln(x) = \frac{1}{x},$ and use the chain rule)

- $\frac{d}{dx}\log_b(f(x)) =$ $\frac{1}{\ln(b) \cdot f(x)} \cdot f'(x)$
- $\frac{d}{dx}\arccos(x) =$ $-\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\arcsin(x) =$ $\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\arctan(x) =$ $\frac{1}{1+x^2}$

3.9 Related Rates

Learning Objectives: After completing this section, we should be able to

- solve related rates problems in various real-life situations.

Related rates are all about how multiple rates of change are connected. For example, if I drive north at 15 mph and you drive south at 15 mph, the distance between us is increasing at a rate of 30 mph.
Tips:

- Interpret the derivatives

  as rates of change. Be careful about the sign; i.e., positive or negative

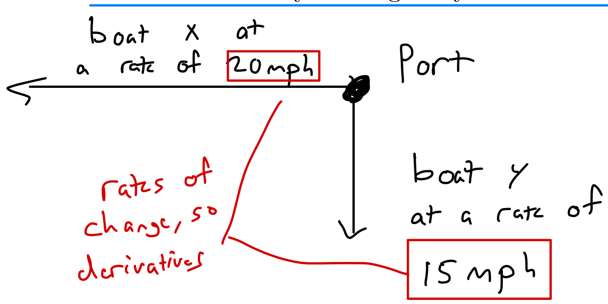
- You may need to come up with an equation (often Pythagorean)

  Draw a picture with labels

- Don't mix up

  $\underbrace{\text{rates}}_{\text{derivatives}}$ and $\underbrace{\text{quantities}}_{}$ typically constant and not measuring a rate of change

Example. Two boats leave a port at 12:00pm. One travels West at 20mph and the other South at 15mph. How fast are they moving away from each other at 1:30pm?



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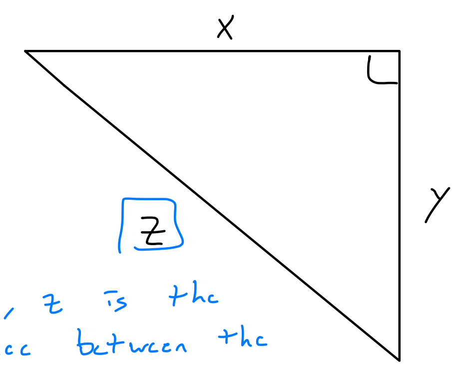


Note, z is the distance between the 2 boats

Want to find the rate of change of z, as this measures how fast the two boats are moving away from each other.

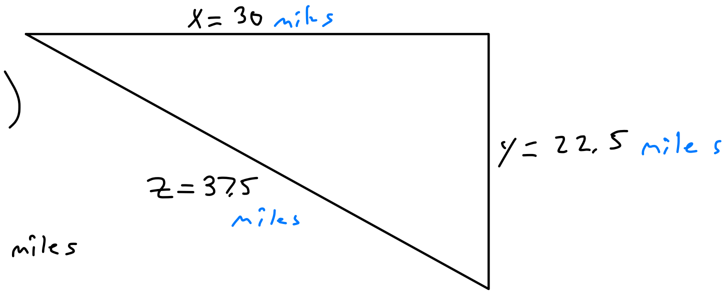
Note, 12:00 pm to 1:30 pm is 1.5 hours

- x is the distance travelled in 1.5 hours by the west-bound boat
- y is the distance travelled in 1.5 hours by the south-bound boat
- z is the distance between the two boats after 1.5 hours
- $\frac{dx}{dt} = 20$ mph => the rate of change of the distance x, i.e., the speed of west-bound boat
- $\frac{dy}{dt} = 15$ mph => the rate of change of the distance y, i.e., the speed of south-bound boat
- $\frac{dz}{dt}$ is the rate of change of the distance between the 2 boats FIND THIS

Example Continued:

$x =$ (rate) (time) $=$ (20 mph) (1.5 hours) $= 30$ miles

$y =$ (rate) (time) $=$ (15 mph)(1.5 hours) $= 22.5$ miles



$x^2 + y^2 = z^2$ (Pythagorean Thm)

$30^2 + (22.5)^2 = z^2$

$\Rightarrow z = \sqrt{30^2 + 22.5^2} = 37.5$ miles

$x = 30$ miles $\quad \frac{dx}{dt} = 20$ mph

$y = 22.5$ miles $\quad \frac{dy}{dt} = 15$ mph

$z = 37.5$ miles $\quad \frac{dz}{dt} = ?$ mph

Need to find rates of change with respect to time

$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}z^2$$

Note, $x, y, \& z$ all depend time implicitly

$$\Rightarrow 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$$

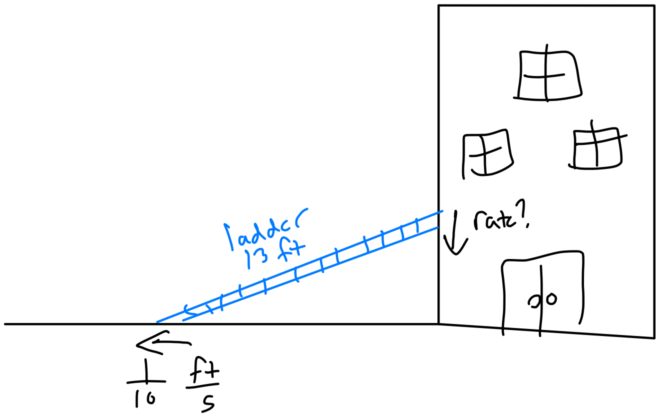
$$\Rightarrow 2(30)(20) + 2(22.5)(15) = 2(37.5)\left(\frac{dz}{dt}\right)$$

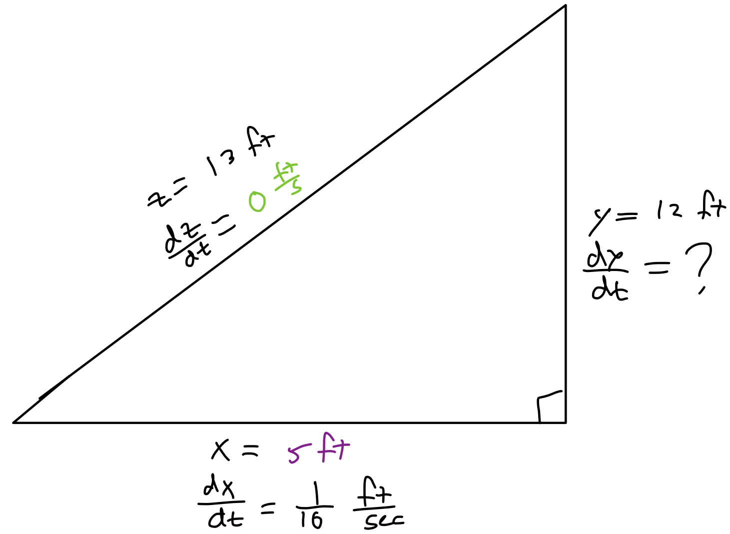
$$\Rightarrow \frac{1875}{75} = \frac{75}{75}\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{1875}{75} = 25 \text{ mph}$$

So, after 1.5 hours, the distance between the 2 boats is changing at a rate of 25 mph

Example. A ladder 13 feet long leans against a building. The base is pulled away from the wall at a rate of $\frac{1}{10}\frac{\text{ft}}{\text{sec}}$. How fast is the top moving down when the top is 12 feet above ground?





Note, $\frac{dz}{dt} = 0\ \frac{\text{ft}}{\text{sec}}$, as the length of the ladder is constantly 13 ft

$$x^2 + y^2 = z^2$$
$$\Rightarrow x^2 + 12^2 = 13^2$$
$$\Rightarrow x = \sqrt{13^2 - 12^2} = 5 \text{ ft}$$

Quantities	Rates
$x = 5$ ft	$\frac{dx}{dt} = \frac{1}{10}\ \text{ft/sec}$
$y = 12$ ft	$\frac{dy}{dt} = ?\ \frac{\text{ft}}{\text{sec}}$
$z = 13$ ft	$\frac{dz}{dt} = 0\ \frac{\text{ft}}{\text{sec}}$

$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}z^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$2(5)\left(\frac{1}{10}\right) + 2(12)\frac{dy}{dt} = 2(13)(0)$$
$$1 + 24\frac{dy}{dt} = 0$$
$$\Rightarrow 24\frac{dy}{dt} = -1 \Rightarrow \frac{dy}{dt} = -\frac{1}{24}\ \frac{\text{ft}}{\text{sec}}$$

build in the minus sign

The top of the ladder is moving down the building at a rate of $\frac{1}{24}\ \frac{\text{ft}}{\text{sec}}$

You try!

Example. Suppose you are blowing up a spherical balloon at a rate of 3 $\frac{\text{cm}^3}{\text{s}}$. How fast is the radius of the balloon changing when $r = 5$ cm? Volume of sphere, $V = \frac{4}{3}\pi r^3$

Rates in problem:

$\frac{dV}{dt} = 3\,\frac{cm^3}{s}$

$\frac{dr}{dt} = ?$ Find this!

$$\frac{d}{dt}V = \frac{d}{dt}\frac{4}{3}\pi r^3$$

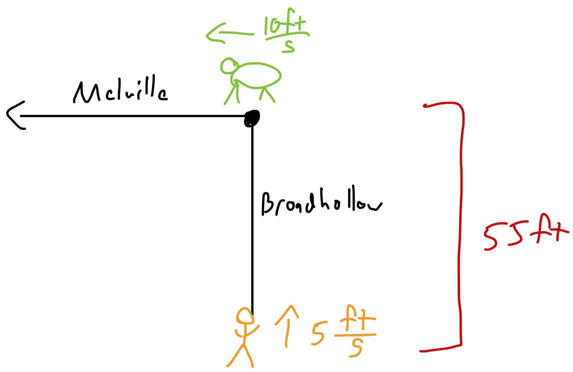
$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}r^3 = \frac{4}{3}\pi\, 3r^2\frac{dr}{dt}$$

$$\Rightarrow 3\left(\frac{cm^3}{s}\right) = \frac{4}{3}\pi\cdot 3\cdot(\cancel{3}\ 5\ cm)^2\cdot\frac{dr}{dt}$$

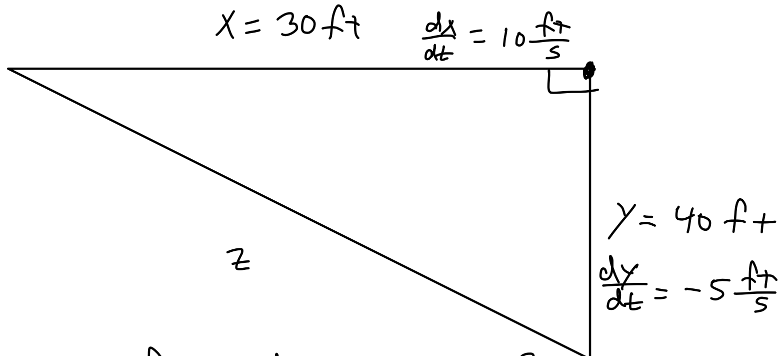
$$\Rightarrow \frac{dr}{dt} = \frac{3\left(\frac{cm^3}{s}\right)}{\frac{4}{3}\pi\, 3(\cancel{3}\ 5\,cm)^2} = \frac{3}{\frac{4}{3}\cdot\pi\cdot 3\cdot 5^2}\,\frac{cm}{s}$$

Example. John Nader is quickly walking north on Broadhollow Road at 5 feet per second while watching Rambo the Ram trotting west on Melville Road at 10 feet per second. At the moment, Rambo is at the intersection of Melville and Broadhollow and John Nader is 55 feet away from the same intersection.

After three seconds, at what rate is the distance between ~~Marvin~~ Dr. Nader and ~~Blaster~~ Rambo increasing? Remember to use accurate units!



After 3 seconds



After 3 seconds, how far is Dr. Nader from the intersection?

$$55\ ft - \left(5\,\frac{ft}{s}\right)(3\ s) = 40\ feet$$

After 3 seconds, how far is Rambo from the intersection?

$$0\ ft + \left(10\,\frac{ft}{s}\right)(3\ s) = 30\ ft$$

After 3 seconds, how far apart are Dr. Nader and Rambo?

$$x^2 + y^2 = z^2$$

$$\Rightarrow (30\,ft)^2 + (40\,ft)^2 = z^2$$

$$\Rightarrow z = \sqrt{30^2 + 40^2}\ ft = 50\ ft$$

Recap:

Distances

$x = 30$ ft

$y = 40$ ft

$z = 50$ ft

Rates

$\frac{dx}{dt} = 10\,\frac{ft}{s}$

$\frac{dy}{dt} = \cancel{10}\ -5\,\frac{ft}{s}$

$\frac{dz}{dt}$ = rate of change in the distance between Rambo and Dr. Nader

$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}z^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow 2(30\ ft)\left(\cancel{-5\ ft}\ 10\,\frac{ft}{s}\right) + 2(40\ ft)\left(-5\,\frac{ft}{s}\right) = 2\cdot(50\ ft)\,\frac{dz}{dt}$$

$$\Rightarrow \left(2\cdot 30\cdot 10 + 2\cdot 40\cdot(-5)\ \frac{ft^2}{s}\right) = (100\ ft)\,\frac{dz}{dt}$$

$$\Rightarrow \frac{\left(2\cdot 30\cdot 10 + 2\cdot 40\cdot(-5)\ \frac{ft^2}{s}\right)}{(100\ ft)} = \frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{2\cdot 30\cdot 10 + 2\cdot 40(-5)}{100}\ \frac{ft}{s} = 2\ \frac{ft}{s}$$

After 3 seconds, the distance between Dr. Nader and Rambo is increasing at a rate of $2\,\frac{ft}{s}$